Group Theory and Its Applications in Quantum Mechanics

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A Report

On

Group Theory

BY

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2010A3B5174G

(B.E EEE + M.Sc.Physics)

Submitted TO

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ACKNOWLEDGEMENTS

I wish to express my deep sense of gratitude to my project supervisor ***Dr. V Sunil Kumar*** *for his able guidance and useful suggestions, which helped me in understanding the basics of the Group Theory. He has been highly supportive and considerate while dealing with my mistakes.*

I would also like to thank Dr. Chandradew Sharma, for spending his valuable time to explain certain concepts of Quantum Mechanics and their relation with group theory.

Kushagra Nigam

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1. Introduction to group theory

Group theory is a study of mathematical structures known as groups. It aims to classify physical systems as groups and exploits their symmetries to study the effect of various operators acting on that system. Based on the results one can classify these operators or transformations into classes and deal with them all at once rather than dealing with them separately. The concepts of group theory can be applied to establish a one to one isomorphic mapping between two systems. Once this is done, we don’t need to analyze them separately. Rather, we can choose a simpler system and map the result of analysis by transformation matrices.

This report aims to analyze the basic concepts of group theory, various types of matrices and the representation of groups.. Besides this, some important proofs that form the basis of group theory have been taken up in the appendix.

2. Types of matrices

Before I start the proofs of basic theorems involved behind group theory, I would like to summarize various types of matrices in this section:

Types of Matrices

1) **Matrices in real domain:**

Row Matrix

A row matrix is formed by a single row.

$$\begin{pmatrix} 2 & 3 & -1 \end{pmatrix}$$

Column Matrix

A column matrix is formed by a single column.

$$\begin{pmatrix} -7 \\ 1 \\ 6 \end{pmatrix}$$

Rectangular Matrix

A rectangular matrix is formed by a different number of rows and columns, and its dimension is noted as: **mxn**.

$$\begin{pmatrix} 1 & 2 & 5 \\ 9 & 1 & 3 \end{pmatrix}$$

Square Matrix

A square matrix is formed by the same number of rows and columns.

The elements of the form $\mathbf{a_{ii}}$ constitute the principal diagonal.

The secondary diagonal is formed by the elements with $\mathbf{i+j = n+1}$.

$$\begin{pmatrix} 1 & 2 & -5 \\ 3 & 6 & 5 \\ 0 & -1 & 4 \end{pmatrix}$$

Zero Matrix

In a zero matrix, all the elements are zeros.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Upper Triangular Matrix

In an upper triangular matrix, the elements located below the diagonal are zeros.

$$\begin{pmatrix} 1 & 7 & -2 \\ 0 & -3 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

Lower Triangular Matrix

In a lower triangular matrix, the elements above the diagonal are zeros.

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{pmatrix}$$

Diagonal Matrix

In a diagonal matrix, all the elements above and below the diagonal are zeros.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Scalar Matrix

A scalar matrix is a diagonal matrix in which the diagonal elements are equal.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Identity Matrix

An identity matrix is a diagonal matrix in which the diagonal elements are equal to 1.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Transpose Matrix

Given matrix A, the transpose of matrix A is another matrix where the elements in the columns and rows have switched. In other words, the rows become the columns and the columns become the rows.

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{pmatrix} \qquad A^t = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\mathbf{(A^t)^t = A}$$

$$\mathbf{(A + B)^t = A^t + B^t}$$

$$\mathbf{(\alpha \cdot A)^t = \alpha \cdot A^t}$$

$$\mathbf{(A \cdot B)^t = B^t \cdot A^t}$$

Regular Matrix

A regular matrix is a square matrix that has an inverse.

Singular Matrix

A singular matrix is a square matrix that has no inverse.

Idempotent Matrix

The matrix A is idempotent if:

$$\mathbf{A}^2 = \mathbf{A}.$$

Involutive Matrix

The matrix A is involutive if:

$$\mathbf{A}^2 = \mathbf{I}.$$

Symmetric Matrix

A symmetric matrix is a square matrix that verifies:

$$\mathbf{A} = \mathbf{A}^t.$$

Antisymmetric Matrix

An antisymmetric matrix is a square matrix that verifies:

$$\mathbf{A} = -\mathbf{A}^t.$$

Orthogonal Matrix

A matrix is orthogonal if it verifies that:

$$\mathbf{A} \cdot \mathbf{A}^t = \mathbf{I}.$$

After studying matrices in real domain, I would like to take up various types of matrices in complex domain that presents a much more general view of real domain and play a central role in group theory and quantum mechanics.

Note: In following discussion the notation † means adjoint of a matrix that is transpose of conjugate.

2) Matrices in Complex domains

Unitary Matrix

A matrix is unitary if it verifies that:

$$A \cdot A^{\dagger} = I$$

Hermetian Matrix

A matrix is Hermetian if it verifies that:

$$A = A^{\dagger}$$

Normal Matrix

A matrix is Normal if it verifies that:

$$A. A^{\dagger} = A^{\dagger}.A$$

3. Basics of Group Theory

3.1 Postulates of Group

A collection G of operators/elements A,B,C,…. finite or infinite in number, forms a group if it satisfies following mathematical postulates:

i) Closure: Product BA of two elements is an element of the collection G. i.e BA=C C belongs to G
ii) The identity E is a part of the collection that satisfies AE=EA=A.
iii) To each element A corresponds another operation A^{-1}, called the inverse of A, belonging to G and given by:
$$AA^{-1} = A^{-1}A = E$$
iv) Associativity: The product of these operations satisfies associative law i.e. (AB)C=A(BC)

A group whose elements satisfy commutative law is known as **Abelian**. If number g of operations belonging to G is finite, the group is finite and g is the order.

Some examples of group:

i) A collection or set of rational numbers form a group under operation of multiplication.
ii) Set of all integers form a group under addition form a group.

3.2 Subgroups

A finite/infinite set H of operations, in group G is its subgroup if it satisfies following conditions:

i) Products of two of the operations belong to H
ii) If H contains the elements A,B,C.. it should also contain their inverses and hence an identity element.

A subgroup is said to be invariant which allows the same operators as the main group.

3.3 Classes or complexes

Two elements A and B of a group are said to be conjugated if one can find a third element S of the same group such that

$$B = SAS^{-1}$$

A class is a collection of all the operations conjugate to a given operation.

Results: - The trace of elements belonging to same class is same.

3.4 Cosets

Let's take an invariant subgroup H of G consisting of elements A,B,C….. Now we take an element s_2 of G that does not belong to H and form the product s_2A, s_2B, s_2C … This collection constitutes complex of elements or left coset as it is assocated at the left of the subgroup H indicated by s_2H.

Results: -

i) None of s_2A, s_2B, s_2C … elements belong to H because if they would, then by definition of subgroup $s_2AA^{-1} = s_2$ would also belong to H, which is not true.
ii) These elements do not form a subgroup as they do not possess identity element. Moreover, s_2As_2B is not a part of subgroup as As_2B does not belong to H.

Similarly if we take an element s_3 that doesn't belong to H and s_2H then we can form a second coset s_3H. Hence, exhausting this procedure, we can reconstruct the entire group by summing up all the cosets.

Hence, $G = H + s_2H + s_3H + s_4H \ldots\ldots s_lH$

Each of the coset contains same number h, of different elements as H, hence we have g=lh.

The order of a subgroup of a finite group is an integral divisor of order of group. The integer l is the index of subgroup.

3.5 Factor Group

Let's again consider an invariant subgroup H of G and decompose G into its cosets such that:

$$G = H + s_2H + s_3H + s_4H \ldots.. s_lH$$

Now, each term in this sum can be considered as an element of a new group known as factor group of G and is designated by G/H.

If we take the product s_iHs_jH we can show that this is another coset of the form s_kH associated with H.

$$s_iHs_jH = s_is_js_j^{-1}Hs_jH$$

Since, H is invariant, $s_j^{-1}Hs_j$ is nothing but H itself.

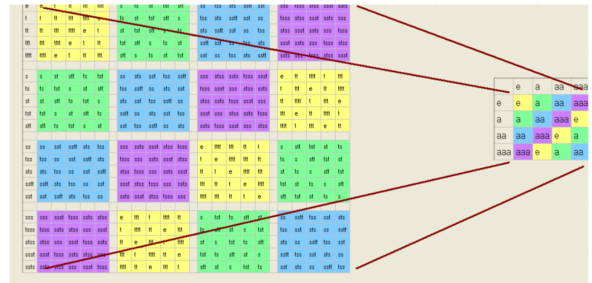
Hence, $s_is_js_j^{-1}Hs_jH = s_is_jH = s_kH$ which is nothing but a new coset and hence an element of factor group.

Hence designating $H \; with \; F_1, \; s_2H \; with \; F_2, \; s_3H \; with \; F_3 \; and \; so \; on$, we can consider F_1, F_2, F_3 to be the elements of factor group.

Hence factor group is a very fine example of many to one homomorphism from the elements of a coset to a single element F.

In case of abelian group as AS=SA for every S. Hence, each subgroup being invariant can be used for defining a factor group.

3.6 Isomorphism and Homomorphism



Given any two groups (G, *) to (H, ^{o}), a

Group homomorphism from (G, *) to (H, ^{o}) is a function

h: G -> H $\forall\, u, v \in G$ *it holds that*
h(u * v) = h(u).h(v)

h-> maps the identity element e_G of G to identity element e_H of H and it also maps inverse s.t. $h(s^{-1}) = h(s)^{-1}$

Proof: -

i) h(a * e_G) = h(a).h(e_G)
$\Rightarrow \mathrm{h(\,a)} = \mathrm{h(a).h}(e_G)$
$\Rightarrow e_H = h(e_G)$

ii) h(a *a^{-1}) = h(a).h(a^{-1})
$\Rightarrow \mathrm{h}(e_G) = \mathrm{h(a).h}(a^{-1})$
$\Rightarrow e_H = \mathrm{h(a).h}(a^{-1})$
$\Rightarrow \mathrm{h(a)}^{-1} = \mathrm{h}(a^{-1})$

Note: Proof # 3 given in appendix illustrates a relation between isomorphism, homorphism and factor groups.

3.7 Representations of Group

If we map an arbitrary group G homomorphically on a group of operators D(G) in vector space L, we say that the operator group D(G) is a representation of the group G in representation space L. If the dimensionality of L is n then the degree of the representation is n.

The operator corresponding to R of G is denoted by D(R). If R and S are elements of group G, then

$D(RS) = D(R)D(S)$

$D(R^{-1}) = [D(R)]^{-1}$

$and\ D(E) = 1$

3.8 Equivalent Representations

If we change the basis in n-dimensional space L, the matrices D(R) will be replaced by their transforms by some matrix C given by,

D(R)= C D(R)C^{-1} also provide a representation of group G, which is equivalent to the representation D(R).

Given two equivalent representations of a group G, we can verify that they share common trace denoted by X(R).

$$X(R) = \sum_{ii} D(R)$$

Thus the trace is invariant under transformation of coordinate axis and in case of group representation it is known as character of representation.

Hence, we see that equivalent representations have same set of characters and hence are denoted by X^{μ}(R) or [μ;R]. This means character of R in μ representation space.

Thus one can relate classes of a group with the characters. If a group is classified in to classes $K_1, K_2, K_3, \ldots., K_n$ then the representation can be described with a set of characters $X_1, X_2, X_3, \ldots. X_n$, where n are the number of classes. For different representations one can use the superscript μ i.e $X_1{}^{\mu}, X_2{}^{\mu}, X_3{}^{\mu}, \ldots. X_n{}^{\mu}$ describes characters in $\mu\ representation.$

4. Continuous Groups And their Applications

Most of the part covered in this report uphill now was related to finite groups. These find immense applications in spectroscopy where one needs to analyze molecules and their spectra.

In this section we will talk about continuous groups and their applications in quantum mechanics.

4.1 Continuous Groups

Whenever we consider the order of a group to be infinite they constitute what is called as lie groups. They find their utility in analyzing symmetries of continuous objects.

As an example, one can consider SO(3) as a group of symmetries of a sphere.

4.2 Application in Quantum Mechanics

Symmetry groups and Lie algebras appear throughout physics, but it is in quantum mechanics that their presence is most obvious. This is because the structure of the theory is such that it can be formulated entirely in a linear algebra context, where we have a vector space V and we are concerned with observables $O \epsilon \text{End}V = gl(V)$, which is also the space in which all representations lie.

4.3 Group of 2-d Isotropic Harmoics Oscillator

Here, we will take an example of 2-D Isotropic Harmonic Oscillator to illustrate the application of lie algebra.

We first look for quantities that are conserved in 2-D isotropic harmonic oscillator i.e. we seek quantities quadratic in canonical coordinates and momenta that commute with Hamiltonian.

Let's first define some part of Hamiltonian dynamics.

Consider a conservative system of particles with N generalized coordinates $\vec{q}$ and their conjugate momenta $\vec{p}$. Then this system is described by some Hamiltonian H (q1::qN; p1::pN), and the equations of motion are:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\dot{q}_i = -\frac{\partial H}{\partial p_i}$$

Using these equations of motion, we can find the equations of motion for any given observable O(q1…qN,p1…pN):

$$\dot{\boldsymbol{O}} = \sum_{\boldsymbol{i=1}}^{\boldsymbol{N}} \left(\frac{\partial H}{\partial q_i}\frac{\partial O}{\partial p_i} - \frac{\partial O}{\partial q_i}\frac{\partial H}{\partial p_i}\right)$$

The generalized operation is written in terms of Poisson Bracket

$$\{\boldsymbol{A}, \boldsymbol{B}\} = \sum_{\boldsymbol{i=1}}^{\boldsymbol{N}} \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q_i}\frac{\partial A}{\partial p_i}\right)$$

Due to the linearity of derivative, the bracket is bilinear in nature and it is antisymmetric under A <-> B.

Now let us consider O as our conserved quantity which satisfies commutation relation with hamiltoian under poisson bracket i.e.

$$\dot{O} = \{H, O\} = 0$$

Where $H = \frac{{p_x}^2 + {p_y}^2}{2} + \frac{x^2 + y^2}{2}$

where, $\vec{x} \rightarrow \sqrt{m} w_o \vec{x}$

and $\vec{p} \rightarrow \frac{1}{\sqrt{m}} \vec{p}$

Now, as the potential depends only on $r^2 = x^2 + y^2$, we have angular momentum conserved.

Now Hamiltonian can be expressed completely in to Hx and Hy where, these are x and y components of Hamiltonian.

Considering the two constants of motion as A and B where,

$\{H, \{A, B\}\} = -\{A, \{B, H\}\} - \{B, \{H, A\}\}$

$= -\{A, 0\} - \{B, 0\}$

$= 0$

By following Jacobi identity,

[x[yz]] + [y[zx]] + [z[xy]] = 0

If A and B are constants of motion then,

{A,B} commutator is alos a constant of motion.

Therefore our three constants of motion are,

$\{xp_y - yp_x, \frac{{p_x}^2 + {p_y}^2}{2} - \frac{x^2 + y^2}{2}\} = p_x p_y + xy$

Denoting these as

$$L_z = xp_y - yp_x$$

$$L_+ = p_x p_y + xy$$

$$L_- = \frac{{p_x}^2 + {p_y}^2}{2} - \frac{x^2 + y^2}{2}$$

And these satisfy SO(3) algebra,

$$\{L_z, L_\pm\} = \pm 2L_\mp$$

$$\{L_+, L_-\} = 2L_z$$

Let's consider quadratic combinations of coordinates and momenta. One of these combinations is known as routhian given by

$$R_{\pm} = xp_x \pm yp_y$$

$$\{H, R_{\pm}\} = -2V_{\pm} = -2(\frac{{p_x}^2 - x^2}{2} \pm \frac{{p_y}^2 - y^2}{2})$$

$$\{H, V_{\pm}\} = 2R_{\pm}$$

$$\{V_{\pm}, R_{\pm}\} = -2H$$

Results:

i) **Subalgebra commuting with Hamiltonian is not SO(2) (rotations in two dimensions) but SO(3) (rotations in three dimensions).**

ii) **Considering further combinations of the generators we have,**

$$\boldsymbol{L^2 = {L_z}^2 + {L_+}^2 + {L_-}^2 = H^2}$$

$$\boldsymbol{H^2 - {R_z}^2 - {V_z}^2 = {L_z}^2}$$

$$\boldsymbol{H^2 - {R_-}^2 - {V_-}^2 = {L_-}^2}$$

$$\boldsymbol{H^2 - {R_+}^2 - {V_+}^2 = {L_+}^2}$$

$$\boldsymbol{2H^2 - \sum R^2 - \sum V^2 = 0}$$

Conclusion

Group theory finds vast applications in the field of science. It can reduce the enormous computations involved in analyzing physical systems by exploiting their symmetries. Hence, most of my study was based on strengthening the theoretical aspects of group theory rather than going for its applications. In the semesters to come I will try to concentrate more on application part esp. in theory of relativity, quantum mechanics and spectroscopy some part of which I have done during the analysis of various types of symmetries present in the molecules.

Appendix

Proof 1: Hermetian Matrices

Hermetian matrices play a very important role in quantum mechanics as they represent the Eigen values of quantum mechanical states when diagonalized.

Considering this fact I would like to take up some important proofs regarding the Hermetian matrices.

Proof 1.1:

Statement: - Eigen values of Hermetian matrices are necessarily real.

Proof: - In quantum mechanics we define Hermetian adjoint as:

$< A\vec{v}, \vec{w} > = < \vec{v}, A^{\dagger}\vec{w} >$ where A is Hermetian matrix and $\vec{v}, \vec{w} \epsilon V$

Since,

$A^{\dagger} = A$

We have,

$< A\vec{v}, \vec{w} > = < \vec{v}, A\vec{w} >$

Now, this must be true for any vectors $\vec{v}, \vec{w} \epsilon V$. Hence, there is no harm in choosing them to be equal and an eigen vector of matrix A; i.e., $A\vec{v} = \lambda\vec{v}$

Therefore,

$< A\vec{v}, \vec{w} > = < \vec{v}, A\vec{w} >$

$\Rightarrow < \lambda\vec{v}, \vec{v} > = < \vec{v}, \lambda\vec{v} >$

$\Rightarrow \bar{\lambda} < \vec{v}, \vec{v} > = \lambda < \vec{v}, \vec{v} >$

Since, $\vec{v} \neq 0$ hence, the inner product can't be 0. Hence, we have to conclude that $\bar{\lambda} = \lambda$ i.e. $\lambda\ is\ real.$

Proof 1.2:

Statement: - Eigenvectors corresponding to non-degenerate eigenvalues of A are orthogonal.

Proof: - To prove this, take $\vec{v}$ and $\vec{w}$ to be eigenvectors of A with corresponding eigenvalues λ and λ'.
As, $< A\vec{v}, \vec{w} > = < \vec{v}, A^{\dagger}\vec{w} >$ where A is Hermetian matrix and $\vec{v}, \vec{w}\ \epsilon\ V$

Since,

$A^{\dagger} = A$

We have,

$< A\vec{v}, \vec{w} > = < \vec{v}, A\vec{w} >$

$\Rightarrow < \lambda\vec{v}, \vec{w} > = < \vec{v}, \lambda'\vec{w} >$

$\Rightarrow \lambda < \vec{v}, \vec{w} > = \lambda' < \vec{v}, \vec{w} >$

$\Rightarrow (\lambda - \lambda^{'}) < \vec{v}, \vec{w} > = 0$

Since, $(\lambda - \lambda^{'}) \neq 0$ hence, the inner product has to be 0. Hence, we have to conclude that $\vec{v}\ and\ \vec{w}\ are\ orthogonal.$

Now we have sufficient tools at hand to study the diagonalization of Hermetian and unitary matrices in unitary space as this plays a central role reduction of representation of a group. The next proof illustrates the conditions required for diagonalization.

Proof 2: Unitary Diagonalization

Statement: - In unitary space every Unitary or Hermetian matrix can be diagonalized by unitary similarity transformation.

Proof: -

Here, I will take up the proof for Hermetian matrix.

Let's take a Hermetian matrix A. One can always find at least one eigenvector and hence a corresponding eigen vector.

Hence, one can always write,

A$\vec{v}_1$ = $\lambda_1\vec{v}_1$, Where, $\vec{v}_1$ is considered to be normalized.

We now construct a unitary matrix U1 as follows. Take the first column of U1 to be given by (the normalized)$\vec{v}_1$. The rest of the unitary matrix will be called Y, which is an n × (n − 1) matrix i.e.

$U_1 = (\vec{v}_1|Y)$

As the columns of U1 comprise an orthonormal set of vectors, we can write the matrix elements of Y in the form $Y_{ij} = (\vec{v}_j)_i$, for i = 1, 2, . . . , n and j = 2, 3, . . ., where $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is an orthonormal set of vectors. Here $(\vec{v}_j)_i$ is the i^{th} coordinate (with respect to a fixed orthonormal basis) of the j^{th} vector of the orthonormal set. It then follows that:

$< \vec{v}_i, \vec{v}_j >= \sum_{k=1}^{n}(\overline{\vec{v}_j})_k(\vec{v}_1)_k = 0 \; for \; j = 2,3, \dots, n$

Where $(\overline{\vec{v}_j})_k$ is the complex conjugate of kth component of vector $\vec{v}_j$.

The above matrix product can be rewritten as

$$Y^\dagger \vec{v}_1 = \sum_{k=1}^{n}(\overline{Y})_{kj}(\vec{v}_1)_k = \sum_{k=1}^{n}\overline{(\vec{v}_j)}_k(\vec{v}_1)_k = 0\,.$$

Computing the product of matrices, we have:

$$U_1^\dagger A U_1 = \left(\begin{array}{c} \vec{v}_1^\dagger \\ \hdashline Y^\dagger \end{array}\right) A \left(\vec{v}_1 \,\vdots\, Y \right) = \left(\begin{array}{c:c} \vec{v}_1^\dagger A \vec{v}_1 & \vec{v}_1^\dagger A Y \\ \hdashline Y^\dagger A \vec{v}_1 & Y^\dagger A Y \end{array}\right) .$$

Using $A\vec{v}_1 = \lambda_1 \vec{v}_1$ with $\vec{v}^\dagger{}_1 \vec{v}_1 = 1$, we have

$$\vec{v}^\dagger{}_1 A \vec{v}_1 = \lambda_1 \vec{v}^\dagger{}_1 \vec{v}_1 = \lambda_1$$

$$Y^\dagger A \vec{v}_1 = \lambda_1 Y^\dagger \vec{v}_1 = 0$$

Hence

$$U_1^\dagger A U_1 = \left(\begin{array}{c:c} \lambda_1 & \vec{v}_1^\dagger A Y \\ \hdashline 0 & Y^\dagger A Y \end{array}\right) .$$

Now as A is Hermetian matrix, one can verify that $(\vec{U}^\dagger{}_1 A \vec{U}_1)^\dagger = \vec{U}^\dagger{}_1 A \vec{U}_1$
Hence, $\vec{U}^\dagger{}_1 A \vec{U}_1$ is also Hermetian.
Thus upper right half of $\vec{U}^\dagger{}_1 A \vec{U}_1$ is also Hermetian. Hence,

$$U_1^\dagger A U_1 = \left(\begin{array}{c:c} \lambda_1 & 0 \\ \hdashline 0 & Y^\dagger A Y \end{array}\right) .$$

In particular, (Y †AY)† = Y †AY . In fact, since U_1†AU_1 is hermitian, it follows that λ_1 is real and Y †AY is hermitian, as expected. Thus, we have reduced the problem to the diagonalization of the (n−1)×(n−1) hermitian matrix Y †AY. Following similar procedure for Y †AY, we arrive at

$$U^\dagger A U = D \equiv \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

where the eigenvalues of A are the diagonal elements of D and the eigenvectors of A are the columns of U. Hence we have diagonalized the hermetian matrix using unitary similarity transformation.

Proof 3: Homomorphism and Factor Groups

Theorem: - If G is homomorphic to G' and letting E' be the unit element of G', then:

1. **The set of elements of G that correspond to E' forms an invariant subgroup of G;**
2. **G' is isomorphic to factor group G/H.**

Proof: -

1) If elements J_1 and J_2 of G correspond to E' in G', then $J_3 = J_1 J_2$ corresponds to E'E'=E'. Hence, the elements J_1, J_2, J_3, J_4…. That correspond to E' forms a subgroup.
   This subgroup is invariant because using an arbitrary element X of G, then XJX^{-1} corresponds in G to $X^{'} E^{'} X^{'-1} = X^{'} X^{'-1} = E'$
   Hence $X^{'} E^{'} X^{'-1}$belongs to H for an arbitrary X.
2) If $s_1 and\ s_2$are two elements of G, not belonging to H and corresponding to same element s' in G', then ${s_1}^{-1} s_2$ corresponds to $s^{'-1} s^{'} = E'$ and belongs to H: ${s_1}^{-1} s_2 = J_m$.Hence $s_2 = J_m s_1$lies in the coset $s_1 H$ associated with H. Hence, two elements $s_1 J_m$ and $s_1 J_n$of this coset correspond to the same element s' in G'. Hence element s' corresponds to coset $s_1\ H$ just as E' corresponds to H. We find G' is isomorphic to factor group G/H.

Proof 4: Unitary Representations

If operators of a group are unitary operators or the matrices of representation are unitary matrices, the representation is said to be unitary representation. There are various advantages of unitary representation as there is always a possibility of reduction of representation of representation matrix as we showed in Proof 2.

Before we take up the proof, here's another definition of unitary matrix in terms of inner product:
A matrix/operator is unitary if <Ux,Uy> = <x,y> $\forall\ x, y$

Statement: - For a finite group every representation is a unitary representation.

Proof: -

For an arbitrary pair of vectors we can redefine the inner product as

$$\{x, y\} = \frac{1}{g} \sum_{R\ c\ G} (D(R)x, D(R)y)$$

The sum runs over all the representations of the group.

$$\{D(S)x, D(S)y\} = \frac{1}{g} \sum_{R\ c\ G} (D(R)D(S)x, D(R)D(S)y)$$
$$= \frac{1}{g} \sum\nolimits_{R\ c\ G} (D(RS)x, D(RS)y)$$

Now for a fixed S, due to closure property as R runs through all the elements of group G, so does RS hence,
one can write,

$\{D(S)x, D(S)y\} = \{x, y\}$ **- i)**

Hence, operators of our representations are unitary with respect to scalar product {x,y}.

Now considering u_i as a set of orthonormal basis vectors with respect to original scalar product and v_i with respect to the new scalar product
i.e. $(u_i, u_j) = \delta_{ij} = \{v_i, v_j\}$.
Considering the operator T which takes u's into the v's:
$v_i = Tu_i$

Now $Tx = Tx_i u_i = x_i T u_i = x_i v_i$

$\{Tx, Ty\} = x_i^* y_i = (x,y)$ **- ii)**

Considering the equivalent representation
$D'(S) = T^{-1}D(S)T$

Now we will try to see if this representation is unitary with respect to original inner product or not.
Considering

$(T^{-1}D(S)Tx, T^{-1}D(S)Ty) = \{D(S)Tx, D(S)Ty\}$ from **ii)**

$= \{Tx, Ty\}$ from **i)**

$= (x, y)$ from **ii)**

Hence, D'(S) is a unitary transformation with respect to original inner product.

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