Neo Classical Limits

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A Report

On

Neo Classical Limits

BY

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(B.E EEE + M.Sc.Physics)

Submitted To

Dr.Raghunath Ratabole

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-Kushagra Nigam

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1. Introduction

The way our thinking evolves from one theory to another is guided by what we already have in hand. Whether it is a theory of big or theory of small, all its formulations cannot escape the beautiful details of existent understanding of scientific community.

Bohr himself formulated Correspondence Principle that states as quoted by wiki "the behavior of systems described by the theory of quantum mechanics (or by the old quantum theory) reproduces classical physics in the limit of large quantum numbers. In other words, it says that for large orbits and for large energies, quantum calculations must agree with classical calculations."

Well if this is true then we certainly cannot move on with an excuse that prohibits a deeper understanding of Quantum Theory.

So, with this motivation, I will try to explore various possible ways to actually link our understanding of Quantum Mechanics with Classical Mechanics.

The first section of this report develops an understanding of Classical Limits of Quantum Harmonic Well in terms of large energy states and Coherent States. Second section carries the motivation of Coherent States to a further develop a mathematical formalism of Vector Models. In the third section, I will use the formalism developed earlier to understand Quantum Oscillators in N dimensions.

2. Quantum Harmonic Oscillator (QHO)

Quantum harmonic oscillators are perhaps the simplest and most widely used system in Quantum Theories. So let's begin by exploring these.

The Hamiltonian for QHO is given as

$$\hat{H} = \frac{1}{2}k\hat{x}^2 + \frac{\hat{p}^2}{2m} \quad \text{where } \wedge \text{ denotes operators} \qquad - (1)$$

The eigenstates of this Hamiltonian are discrete energy states labelled by n that dictates the level of excitation.

$$\therefore \hat{H}|n\rangle = E_n|n\rangle \quad \text{where } E_n \text{ is given as } \left[n+\frac{1}{2}\right]\hbar\omega \qquad - (2)$$

Also, one forms an algebra of two non-Hermetian operators

i) Creation operator $\hat{a}^{+} = \sqrt{\frac{m\omega}{2\hbar}}\left[\hat{x} - \frac{i\hat{p}}{m\omega}\right]$

ii) Annihilation operator $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left[\hat{x} + \frac{i\hat{p}}{m\omega}\right]$

such that

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \qquad - (3)$$

$$\hat{a}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad - (4)$$

also, $\hat{a}^{+}\hat{a} = \hat{N}$ where $\hat{N}|n\rangle = n|n\rangle \qquad - (5)$

Here, $\hat{N}$ is called as number operator.

Now, let us study some of the properties that these states exhibit.

i) $\langle \hat{x} \rangle$: Consider arbitrary $|n\rangle$

$$\langle n|\hat{x}|n\rangle = \langle n|a^{+}+a|n\rangle\sqrt{\frac{\hbar}{2m\omega}}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\langle n|\sqrt{n+1}\,|n+1\rangle + \langle n|\sqrt{n}\,|n-1\rangle\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\,\delta_{n\,n+1} + \sqrt{n}\,\delta_{n\,n-1}\right]$$

$$= 0.$$

ii) $\langle \hat{p} \rangle$:

$$\langle n|\hat{p}|n\rangle = \sqrt{\frac{\hbar m\omega}{2}}\,\frac{\langle n|a^{+}-a|n\rangle}{-i}$$

$$= \sqrt{\frac{\hbar m\omega}{2}}\,\frac{\left[\langle n|\sqrt{n+1}\,|n+1\rangle - \sqrt{n}\langle n|n-1\rangle\right]}{-i}$$

$$= \sqrt{\frac{\hbar m\omega}{2}}\,\frac{\left[\delta_{n\,n+1} - \delta_{n\,n-1}\right]}{-i}$$

$$= 0.$$

iii) $\langle \hat{x}^2 \rangle$:

$$\langle \hat{x}^2 \rangle = \langle n| \hat{a}\hat{a} + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}^\dagger |n\rangle \frac{\hbar}{2m\omega}$$

$$= \frac{\hbar}{2m\omega}\left[\langle n| \hat{a}\hat{a}^\dagger |n\rangle + \langle n|\hat{a}^\dagger\hat{a}|n\rangle\right]$$

$$= \frac{\hbar}{2m\omega}\left[\left[\sqrt{n+1}\right]^2 + n\right] = \frac{\hbar}{2m\omega}\left[2n+1\right] = \frac{\hbar}{m\omega}\left[n+\frac{1}{2}\right]$$

iv) $\langle \hat{p}^2 \rangle$:

$$\langle \hat{p}^2 \rangle = \langle n| \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger - \hat{a}\hat{a} - \hat{a}^\dagger\hat{a}^\dagger |n\rangle \frac{\hbar m\omega}{2}$$

$$= \langle n| \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger |n\rangle \frac{\hbar m\omega}{2} = (2n+1)\frac{\hbar m\omega}{2}$$

$$= \left[n+\frac{1}{2}\right]\hbar m\omega .$$

$$\therefore \langle \Delta\hat{x}^2 \rangle = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2 = \frac{\hbar}{m\omega}\left[n+\frac{1}{2}\right] \quad (6)$$

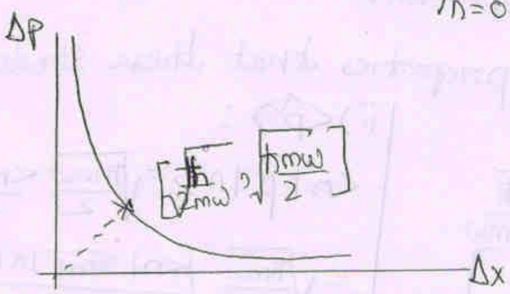
$$\langle \Delta\hat{p}^2 \rangle = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2 = (2n+1)\frac{\hbar m\omega}{2} = \hbar m\omega\left[n+\frac{1}{2}\right] \quad (7)$$

$$\therefore \langle \Delta\hat{x}^2 \rangle \langle \Delta\hat{p}^2 \rangle = \hbar^2\left[n+\frac{1}{2}\right]^2 \quad (8)$$

- What is interesting to see is for $n=0$, we have minimum uncertainty!!

$$\langle \Delta\hat{x}^2 \rangle \langle \Delta\hat{p}^2 \rangle = \frac{\hbar^2}{4}\Bigg|_{n=0}$$

That is, ground state is a minimum uncertainty state.



v) Time evolution of $|n\rangle$:

We know that the time evolution operator is given as:

$$U(t,t_0) = \exp\left[\frac{-iH(t-t_0)}{\hbar}\right] \quad \text{consider } t_0 = 0.$$

$$\therefore U(t) = \exp\left[\frac{-iHt}{\hbar}\right]$$

$$\therefore \quad |n;t\rangle = U(t)|n\rangle = \exp\left[\frac{-iHt}{\hbar}\right]|n\rangle = \exp\left[\frac{-iE_nt}{\hbar}\right]|n\rangle$$

Consider $\langle x|n;t\rangle = \exp\left[\frac{-iE_nt}{\hbar}\right]\langle x|n\rangle$

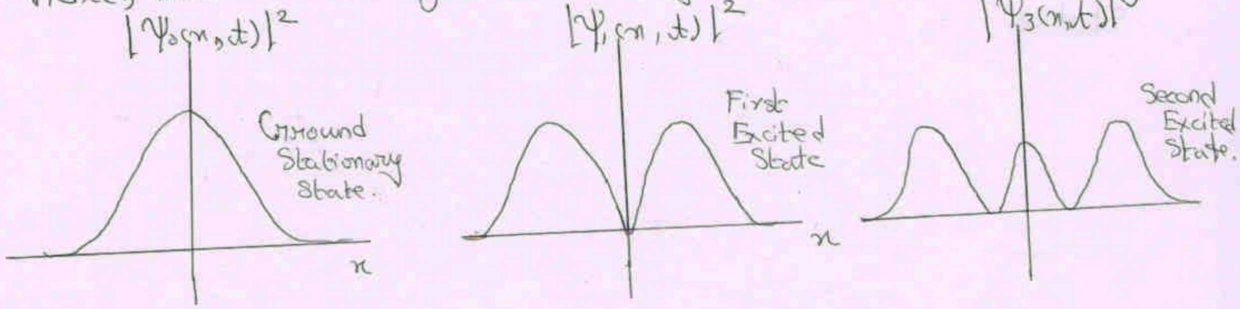
$$= \cos\left[\frac{E_nt}{\hbar}\right]\langle x|n\rangle - i\sin\left[\frac{E_nt}{\hbar}\right]\langle x|n\rangle$$

$$\langle x|n;t\rangle = \cos\left[\left[n+\tfrac{1}{2}\right]\omega t\right]\langle x|n\rangle - i\sin\left[\left[n+\tfrac{1}{2}\right]\omega t\right]\langle x|n\rangle$$

This indicates that real & imaginary part of wavefunction oscillate with time with frequency $\left[n+\frac{1}{2}\right]\omega$

However $|\langle x|n;t\rangle|^2 = |\langle x|n\rangle|^2$ ∴ Amplitude remains stationary.

Hence, the excited & ground states of oscillator are stationary states.



Now consider two state superposition of QHO: $|\psi\rangle = C_n(0)|n\rangle + C_m(0)|m\rangle$

$$\Rightarrow \psi(x,t) = \langle x|C_n(0)|n;t\rangle + \langle x|C_m(0)|m;t\rangle$$

Here, $|\psi;t\rangle = U(t)|\psi\rangle = \exp\left[\frac{-iHt}{\hbar}\right]\left[C_n(0)|n\rangle + C_m(0)|m\rangle\right]$

$$= C_n(0)\exp\left[\frac{-iE_nt}{\hbar}\right]|n\rangle + C_m(0)\exp\left[\frac{-iE_mt}{\hbar}\right]|m\rangle$$

$$\begin{aligned}
\langle \hat{x}\rangle &= \langle \psi;t|\hat{x}|\psi;t\rangle \\
&= \langle n|x|n\rangle|C_n(0)|^2 + \langle m|x|m\rangle|C_m(0)|^2 \\
&\quad + \langle n|x|m\rangle C_n^*(0)C_m(0)\exp\left[\frac{it}{\hbar}[E_m-E_n]\right] \\
&\quad + \langle m|x|n\rangle C_n(0)C_m^*(0)\exp\left[\frac{i}{\hbar}t[E_n-E_m]\right] \\
&= \langle n|x|m\rangle C_n^*(0)C_m(0)\exp\left[\frac{it}{\hbar}[E_m-E_n]\right] \\
&\quad + \left[\langle n|x|m\rangle C_n^*(0)C_m(0)\exp\left[\frac{it}{\hbar}[E_m-E_n]\right]\right]^* \\
&= 2\,\mathrm{Re}\left[\langle n|x|m\rangle C_m(0)C_n^*(0)\exp\left[\frac{it}{\hbar}[E_m-E_n]\right]\right]
\end{aligned}$$

Points that can be extracted:

i) Expectation value is real as expected.

ii) $\langle \hat{x} \rangle$ oscillates back & forth with frequency $\propto (E_n - E_m)$ i.e. difference of energies.

iii) Such states are also called as "Non-Stationary States".

iv) For such states, the amplitude $|\psi(x,t)|^2$ changes with time, mean oscillates but variance is still fixed.

With this much understanding of Quantum Harmonic Oscillators, we will move forward to Coherent states which are again minimum uncertainty states.

3. Coherent State Formulation of QHO

Till now, we have seen stationary & nonstationary states of QHO & I also showed that vaccum state is minimum uncertainty state.

Now, let us see the same result from a different angle:

$$\hat{a}|0\rangle = 0 \quad - \quad (9)$$

$\hat{a}$ ↑ Annihilation. 0 ↑ Scalar.

Taking component of $\hat{a}|0\rangle$ along $\langle x'|$ bra, we have

$$\langle x'|\hat{a}|0\rangle = 0$$

$$\Rightarrow \langle x'|\sqrt{\frac{m\omega}{2\hbar}}\left[\hat{x} + \frac{i\hat{p}}{m\omega}\right]|0\rangle = 0$$

$$\Rightarrow \langle x'|\sqrt{\frac{m\omega}{2\hbar}}\,\hat{x}|0\rangle + \langle x'|\sqrt{\frac{m\omega}{2\hbar}}\,\frac{i\hat{p}}{m\omega}|0\rangle = 0$$

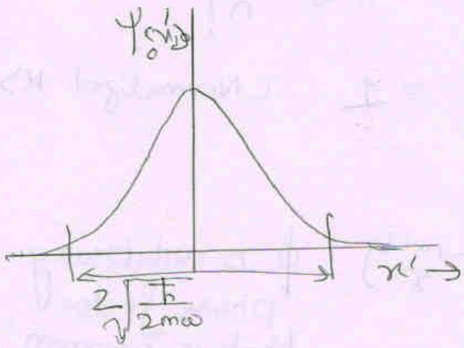
$$\Rightarrow x'\langle x'|0\rangle + \frac{\hbar}{m\omega}\frac{d}{dx'}\langle x'|0\rangle = 0$$

Solving this DE for $\langle x'|0\rangle$ & normalizing, we have

$$\langle x'|0\rangle = \frac{1}{\pi^{1/4}\sqrt{x_0}}\left[\exp\left[-\frac{1}{2}\left[\frac{x'}{x_0}\right]^2\right]\right] = \psi_0(x')$$

where $x_0 = \sqrt{\frac{\hbar}{m\omega}}$

But this is a ground state wavefunction which is a gaussian distribution with mean $\mu_x = 0$ & variance $\sigma_x^2 = \frac{\hbar}{2m\omega}$!!



But one can at the same time write the following relation trivially:

$$a|0\rangle = 0|0\rangle = 0$$

$\therefore$ $|0\rangle$ is an eigenstate of a. But what if I take a general state $|\alpha\rangle$ s.t. $a|\alpha\rangle = \alpha|\alpha\rangle$ will it be a minimum uncertainty state?

Well Let's check!

Consider, $a|\alpha\rangle = \alpha|\alpha\rangle$ (α is complex as $a^\dagger \neq a$)

$\therefore \quad |\alpha\rangle = \sum_{n=0}^{\infty} C_n |n\rangle$

$a|\alpha\rangle = \sum_n C_n\, a|n\rangle = \sum_n C_n \sqrt{n}\, |n-1\rangle = \alpha|\alpha\rangle$

$\Rightarrow \sum_n C_n \sqrt{n}\, |n-1\rangle = \sum_n \alpha C_n |n\rangle$

$\therefore \left[\sum_n C_{n+1}\sqrt{n+1} - \sum_n \alpha C_n\right]|n\rangle = 0.$

$\Rightarrow \sum_n \left[C_{n+1}\sqrt{n+1} - \alpha C_n\right]|n\rangle = 0$

$$\therefore \quad C_{n+1} = \frac{\alpha C_n}{\sqrt{n+1}} \qquad — (10)$$

$C_1 = \frac{\alpha}{\sqrt{1}} C_0$

$C_2 = \frac{\alpha^2}{\sqrt{2!}} C_0$

$\vdots$

$C_n = \frac{\alpha^n}{\sqrt{n!}} C_0$

Thus, $C_n = \frac{\alpha^n C_0}{\sqrt{n!}} \Rightarrow |\alpha\rangle = \sum_n \frac{\alpha^n C_0}{\sqrt{n!}} |n\rangle$

Normalizing:

$$\langle\alpha|\alpha\rangle = \sum_n \sum_m \frac{\alpha^{*m} \alpha^n C_0^* C_0 \langle m|n\rangle}{\sqrt{m!\,n!}}$$

$$= \sum_n \sum_m \frac{\alpha^{*m} |C_0|^2 \delta_{mn} \alpha^n}{\sqrt{m!\,n!}} = \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}\, |C_0|^2}{n!}$$

$$= |C_0|^2 \exp\left[|\alpha|^2\right] = 1 \qquad (\text{Normalized } |\alpha\rangle)$$

$\therefore \quad |C_0|^2 = \exp\left[-|\alpha|^2\right]$

$\Rightarrow |C_0| = \exp\left[-\frac{|\alpha|^2}{2}\right]$

$\therefore C_0 = \exp(i\phi)\exp\left(-\frac{|\alpha|^2}{2}\right)$ $\quad \phi$ is arbitrary phase factor that is "common" to all C_n's due to recursion (10).

$\therefore$ taking $\phi = 0$

$C_0 = \exp\left[\frac{-|\alpha|^2}{2}\right] \Rightarrow \boxed{|\alpha\rangle = \exp\left[\frac{-|\alpha|^2}{2}\right]\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle}$

Thus, we have found coherent state $|\alpha\rangle$

$$|\alpha\rangle = \exp\left[-\frac{|\alpha|^2}{2}\right] \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad -(11)$$

Let us again study all the properties that we did for SHO:

i) $\langle \hat{x} \rangle$:

$$\langle \alpha | \hat{x} | \alpha \rangle = \langle \alpha | \left[\sqrt{\frac{m\omega}{2\hbar}}\right]^{-1} (\hat{a}^\dagger + \hat{a}) |\alpha\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \sum_{n,m} \langle n | \frac{\alpha^{*n}}{\sqrt{n!}} (\hat{a}^\dagger + \hat{a}) \frac{\alpha^m}{\sqrt{m!}} |m\rangle \exp\left[-|\alpha|^2\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \exp\left[-|\alpha|^2\right] \sum_{n,m} \frac{\alpha^{*n}\alpha^m}{\sqrt{n!\,m!}} \Big[\underbrace{\langle n | m+1\rangle}_{\delta_{n\,m+1}} \sqrt{m+1} + \underbrace{\langle n | m-1\rangle}_{\delta_{n\,m-1}} \sqrt{m} \Big]$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \exp\left[-|\alpha|^2\right] \left[\sum_m \frac{\alpha^{*m+1}\alpha^m \sqrt{m+1}}{\sqrt{(m+1)!\,m!}} + \sum_m \frac{\sqrt{m}\,\alpha^{*m-1}\alpha^m}{\sqrt{(m-1)!\,m!}} \right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \exp\left[-|\alpha|^2\right] \left[\sum_m \frac{\alpha^{*m+1}\alpha^m}{m!} + \sum_m \frac{\alpha^{*m}\alpha^{m+1}}{m!} \right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \exp\left[-|\alpha|^2\right] \left[\sum_m \frac{[\alpha^*\alpha]^m}{m!} \right] [\alpha + \alpha^*]$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \exp\left[-|\alpha|^2\right] \exp\left[|\alpha|^2\right] 2\,\mathrm{Re}(\alpha)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\, 2\,\mathrm{Re}(\alpha) = \sqrt{\frac{2\hbar}{m\omega}}\, \mathrm{Re}(\alpha)$$

$$\Rightarrow \mathrm{Re}(\alpha) = \sqrt{\frac{m\omega}{2\hbar}} \langle \hat{x} \rangle \quad -(12)$$

ii) $\langle \hat{p} \rangle$:

Similarly,

$$\langle \hat{p} \rangle = i\sqrt{\frac{m\omega\hbar}{2}} [\alpha^* - \alpha] = 2\,\mathrm{Im}(\alpha) \sqrt{\frac{m\omega\hbar}{2}}$$

$$\Rightarrow \mathrm{Im}(\alpha) = \sqrt{\frac{1}{2m\omega\hbar}} \langle \hat{p} \rangle - (13)$$

iii) $\langle \hat{p}^2 \rangle = \langle \alpha | \hat{p}^2 | \alpha \rangle$

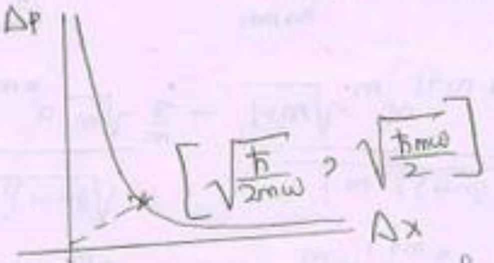
$$= \langle \hat{p} \rangle^2 + \frac{m\omega\hbar}{2}$$

iv) $\langle \hat{x}^2 \rangle = \langle \hat{x} \rangle^2 + \frac{\hbar}{2m\omega}$

Hence, $\langle \Delta \hat{x}^2 \rangle = \frac{\hbar}{2m\omega}$ & $\langle \Delta \hat{p}^2 \rangle = \frac{\hbar m\omega}{2}$

$$\therefore \langle \Delta \hat{x}^2 \rangle \langle \Delta \hat{p}^2 \rangle = \frac{\hbar^2}{4} \; !!$$

Thus, we again have a minimum uncertainty as ~~we~~ predicted in the starting of this section.



Now, let us move a step further and see if coherent states are useful in representing an arbitrary ket of given quantum system of Harmonic Oscillator.

For this we generally want our basis vectors to be
- orthogonal
- complete

Orthogonality basically makes it easier for us to play with mathematics whereas completeness ensures that given set of basis vectors spans the entire "Hilbert space".

So let us test these two properties for $|\alpha\rangle$:

i) • Orthogonality:-

Given $|\alpha\rangle = \exp\left[-\frac{1}{2}|\alpha|^2\right] \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} |m\rangle$

& $|\beta\rangle = \exp\left[-\frac{1}{2}|\beta|^2\right] \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} |n\rangle$

$$\langle \beta | \alpha \rangle = \sum_{m,n} \exp\left[-\frac{1}{2}\left[|\beta^*|^2 + |\alpha|^2\right]\right] \left[\underset{\delta_{nm}}{\underset{\Downarrow}{\langle n | m \rangle}} \frac{\beta^{*n} \alpha^m}{\sqrt{n!m!}} \right]$$

$$= \sum_{m,n} \exp\left[\frac{-1}{2}\left[|\beta|^2 + |\alpha|^2\right]\right] \delta_{nm} \frac{\beta^{*n} \alpha^m}{\sqrt{n!m!}}$$

$$= \sum_n \exp\left[-\frac{1}{2}\left[|\beta|^2+|\alpha|^2\right]\right] \frac{(\alpha\beta^*)^n}{\sqrt{n!n!}}$$

$$= \exp\left[-\frac{1}{2}\left[|\alpha|^2+|\beta|^2\right]+\alpha\beta^*\right] \implies \langle\beta|\alpha\rangle \neq 0 \text{ (Non-orthogonal)}$$

Well, at this point one should not be disappointed as it doesn't rule out their possibility to become basis vectors!!
We just have to show that they are capable of spanning Hilbert space.

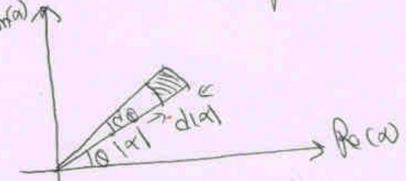
vi) :> <u>Test for completeness:-</u>

Now, testing for completeness of coherent state is little bit tricky as I said earlier α is complex.

Say, $\alpha = Re(\alpha) + i\,Im(\alpha)$

Now, what we need to show for completeness is that $|\alpha\rangle\langle\alpha|$ when integrated over all possible α's give us <u>1</u>.

Consider the following complex plane:



Consider

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\alpha\rangle\langle\alpha|\, d\,Im(\alpha)\, d\,Re(\alpha)$$

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\alpha\rangle\langle\alpha|\, d\,Im(\alpha)\, dRe(\alpha) = \int_0^{2\pi}\int_0^{\infty} |\alpha\rangle\langle\alpha|\, |\alpha|\, d|\alpha|\, d\theta$$

$$= \int_0^{\infty} \sum_{n,m} \frac{\alpha^{*n}\alpha^m}{\sqrt{n!m!}} e^{-|\alpha|^2} |m\rangle\langle n|\; |\alpha|\, d|\alpha|\, d\theta \qquad \left[\alpha = |\alpha|e^{i\theta}\right]$$

$$= \sum_{n,m} \frac{|m\rangle\langle n|}{\sqrt{n!m!}} \int_0^{\infty}\int_0^{2\pi} |\alpha|^n e^{-in\theta}\, |\alpha|^m e^{im\theta}\, |\alpha| d|\alpha| d\theta$$

$$= \sum_{n,m} \frac{|m\rangle\langle n|}{\sqrt{n!m!}} \int_0^{\infty} e^{-|\alpha|^2} |\alpha|^{n+m+1}\, d|\alpha| \int_0^{2\pi} e^{i(m-n)\theta}\, d\theta$$

$$\int_0^{2\pi} e^{i(m-n)\theta} d\theta = \left.\frac{e^{i(m-n)\theta}}{i(m-n)\theta}\right|_0^{2\pi} = 0 \quad \text{if } m \neq n$$

If $m = n$ we have $\int_0^{2\pi} e^{i(m-m)\theta} d\theta = 2\pi$

$$\therefore \int_0^{2\pi} e^{i(m-n)\theta} d\theta = 2\pi\delta_{mn}$$

$$\therefore I = \sum_{n,m} \int_0^\infty \frac{|m\rangle\langle n|}{\sqrt{m!n!}} |\alpha|^{n+m+1} e^{-|\alpha|^2} d|\alpha| \times 2\pi\delta_{mn}$$

$$\cancel{\sum \frac{2\pi}{\sqrt{n!m!}} \int_0^\infty |\alpha|^{2n+1} e^{-|\alpha|^2} d|\alpha| \; |n\rangle\langle m| \delta_{mn}}$$

$$= \sum_{n,m} \frac{|m\rangle\langle n|}{\sqrt{m!n!}} \int_0^\infty |\alpha|^{n+m+1} e^{-|\alpha|^2} d|\alpha| \, 2\pi\delta_{mn}$$

$$= \sum_n \frac{|n\rangle\langle n|}{n!} \int_0^\infty |\alpha|^{2n+1} e^{-|\alpha|^2} d|\alpha| \times 2\pi$$

$$= \sum_n 2\pi \frac{|n\rangle\langle n|}{n!} \int_0^\infty |\alpha|^{2n+1} e^{-|\alpha|^2} d|\alpha|$$

$$= \sum_n \frac{2\pi \, |n\rangle\langle n|}{\cancel{n!}} \times \frac{\cancel{n!}}{2}$$

$$= \sum_n \pi \, |n\rangle\langle n|$$

$$= \pi \;\; !!$$

$$\therefore \int_0^\infty \int_0^{2\pi} |\alpha\rangle\langle\alpha| \, d|\alpha| \, |\alpha| d\theta = \pi > 1$$

Thus, the coherent basis is actually "overcomplete".

- Hence, we can comfortably represent an arbitrary ket in terms of coherent states.

Let us now try to understand how can we use coherent states to understand classical behavior of quantum systems.

4. Classical Characteristics of Coherent States

In this section, I will try to show some of the properties of coherent states that enable us to depict quantum systems on classical "Phase Space".

- Consider coherent state $|\alpha\rangle = \exp\left[\frac{-|\alpha|^2}{2}\right] \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$

I can write $|n\rangle$ as $|n\rangle = \frac{a^\dagger}{\sqrt{n}} |n-1\rangle = \frac{(a^\dagger)^2}{\sqrt{n(n-1)}} |n-2\rangle \cdots = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle$

$$\therefore |\alpha\rangle = \exp\left[-\frac{|\alpha|^2}{2}\right] \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle$$

$$= \exp\left[-\frac{|\alpha|^2}{2}\right] \sum_{n=0}^{\infty} \frac{(\alpha a^\dagger)^n}{n!} |0\rangle$$

$$= \exp\left[-\frac{|\alpha|^2}{2}\right] \exp\left[\frac{\alpha a^\dagger}{1}\right] |0\rangle$$

$$= \exp\left[-\frac{|\alpha|^2}{2}\right] \exp[\alpha a^\dagger] |0\rangle$$

Also, $|0\rangle = \left[1 - \frac{\alpha^* \hat{a}}{1!} + \frac{(\alpha^*\hat{a})^2}{2!} - \frac{(\alpha^*\hat{a})^3}{3!} \cdots\right] |0\rangle$

$$\therefore |\alpha\rangle = \exp\left[\frac{-|\alpha|^2}{2}\right] \exp[\alpha a^\dagger] \exp[-\alpha^* \hat{a}] |0\rangle$$

Consider the following BCH formula: (t is a parameter, $\hat{x}$ & $\hat{y}$ are operators)

$$e^{t(\hat{x}+\hat{y})} = e^{t\hat{x}} e^{t\hat{y}} e^{-\frac{t^2}{2}[\hat{x},\hat{y}]} e^{\frac{t^3}{6}([\hat{y},[\hat{x},\hat{y}]] + [\hat{x},[\hat{x},\hat{y}]])} \cdots\cdots$$

where .... terms contain commutators $[\hat{y},[\hat{x},\hat{y}]]$ & $[\hat{x},[\hat{x},\hat{y}]]$

Now, if $[[\hat{x},\hat{y}],\hat{x}] = [[\hat{x},\hat{y}],\hat{y}] = 0$

$$e^{t(\hat{x}+\hat{y})} = e^{t\hat{x}} e^{t\hat{y}} e^{-\frac{t^2}{2}[\hat{x},\hat{y}]}$$

taking $t=1$, $\hat{x} = \alpha \hat{a}^\dagger$ & $\hat{y} = -\alpha^* \hat{a}$ we see $[\hat{a},\hat{a}^\dagger] = 1$

$$\therefore e^{(\alpha \hat{a}^\dagger - \alpha^* \hat{a})} = \exp\left[-\frac{|\alpha|^2}{2}\right] \exp[\alpha a^\dagger] \exp[-\alpha^* \hat{a}]$$

Hence, $|\alpha\rangle = e^{(\alpha \hat{a}^\dagger - \alpha^* \hat{a})} |0\rangle$

$= D(\alpha)|0\rangle$, where $D(\alpha)$ is known as displacement operator for reasons that will become clear later.

Consider $\alpha = q + ip$

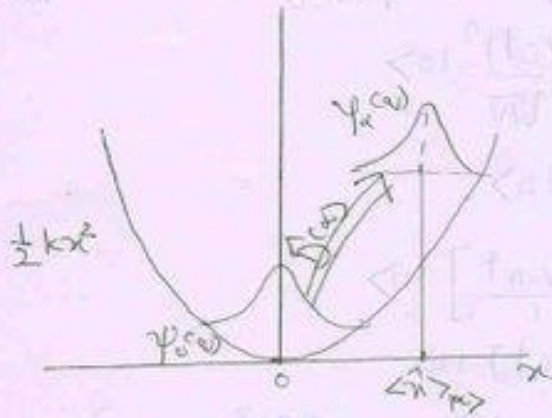
where $q = \mathrm{Re}(\alpha) = \sqrt{\frac{m\omega}{2\hbar}}\, \langle \hat{x} \rangle_{|\alpha\rangle}$

$p = \mathrm{Im}(\alpha) = \sqrt{\frac{1}{2m\omega\hbar}}\, \langle \hat{p} \rangle_{|\alpha\rangle}$

Say, $p = 0$ (as α is arbitrary & q, p are Real)

$\therefore \alpha = q$

$$\therefore \hat{D}(\alpha)|0\rangle = \hat{D}(q)|0\rangle = \exp\left[\alpha \hat{a}^{\dagger} - \alpha^{*}\hat{a}\right]|0\rangle$$
$$= \exp\left[q\hat{a}^{\dagger} - q^{*}\hat{a}\right]|0\rangle$$
$$= \exp\left[q(\hat{a}^{\dagger} - \hat{a})\right]|0\rangle$$
$$= \exp\left[iq\sqrt{\frac{m\omega\hbar}{2}}\sqrt{\frac{2}{m\omega\hbar}}\,\frac{(\hat{a} - \hat{a}^{\dagger})}{i}\right]|0\rangle$$
$$= \exp\left[-iq\,\frac{\hat{p}}{\hbar}\sqrt{\frac{2}{m\omega\hbar}}\,\hbar\right]|0\rangle$$
$$= \exp\left[-\frac{iq}{\hbar}\sqrt{\frac{2\hbar}{m\omega}}\,\hat{p}\right]|0\rangle$$
$$= \underbrace{\exp\left[-\frac{i\langle \hat{x} \rangle_{|\alpha\rangle}\hat{p}}{\hbar}\right]}_{\text{Translation operator in position}}|0\rangle$$



Similarly say $q = 0 \;\therefore\; \alpha = ip$

$$\hat{D}(\alpha)|0\rangle = \hat{D}(ip)|0\rangle = \exp\left[ip\hat{a}^{\dagger} + ip\hat{a}\right]|0\rangle$$
$$= \exp\left[ip(\hat{a}^{\dagger} + \hat{a})\right]|0\rangle$$
$$= \exp\left[\frac{ip\hbar}{\hbar}\sqrt{\frac{2m\omega}{\hbar}}\,\hat{x}\right]|0\rangle$$
$$= \exp\left[\frac{ip}{\hbar}\sqrt{2m\omega\hbar}\,\hat{x}\right]|0\rangle$$
$$= \underbrace{\exp\left[\frac{i\langle \hat{p} \rangle_{|\alpha\rangle}\hat{x}}{\hbar}\right]}_{\text{Translation operator in momentum.}}|0\rangle$$

$\therefore$ As claimed earlier $\hat{D}(\alpha)$ is a displacement operator as it displaces ground state to arbitrary (q, p) state where (q, p) are related to $(\langle \hat{x} \rangle_{|\alpha\rangle}, \langle \hat{p} \rangle_{|\alpha\rangle})$.

• Till now we have shown that coherent states can be displaced from $|0\rangle$ to any arbitrary α.

Let us now see the time evolution of coherent states.

Consider,

$$|\alpha\rangle = \exp\left[\frac{-|\alpha|^2}{2}\right]\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

$$\therefore |\alpha,t\rangle = U(t)|\alpha\rangle = \exp\left[\frac{-|\alpha|^2}{2}\right]\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\exp\left[-\frac{iHt}{\hbar}\right]|n\rangle$$

$$= \exp\left[\frac{-|\alpha|^2}{2}\right]\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\exp\left[\frac{-iE_nt}{\hbar}\right]|n\rangle$$

$$= \exp\left[-\frac{|\alpha|^2}{2}\right]\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\exp\left[-i\frac{\left[n+\frac{1}{2}\right]\hbar\omega t}{\hbar}\right]|n\rangle$$

$$= \exp\left[-\frac{|\alpha|^2}{2}\right]\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\exp\left[-in\omega t\right]\exp\left[\frac{-i\omega t}{2}\right]|n\rangle$$

$$= \exp\left[-\frac{|\alpha|^2}{2}\right]\exp\left[-\frac{i\omega t}{2}\right]\sum_{n=0}^{\infty}\frac{\left[\alpha\exp\left[-i\omega t\right]\right]^n}{\sqrt{n!}}|n\rangle$$

$$= \cancel{\exp\left[-\frac{|\alpha|^2}{2}\right]}\exp\left[\frac{-i\omega t}{2}\right]\frac{|\alpha\exp\left[-i\omega t\right]\rangle}{\cancel{\exp\left[-\frac{|\alpha|^2}{2}\right]}}$$

$$|\alpha,t\rangle = \exp\left[\frac{-i\omega t}{2}\right]|\alpha\exp(-i\omega t)\rangle$$

Hence, $$\langle\hat{x}\rangle = \langle\alpha,t|\,\hat{x}\,|\alpha,t\rangle = \langle\alpha\exp(-i\omega t)|\exp\left[\frac{i\omega t}{2}\right]\hat{x}\exp\left[\frac{-i\omega t}{2}\right]|\alpha\exp(-i\omega t)\rangle$$

$$= \langle\exp(-i\omega t)\alpha|\,\hat{x}\,|\alpha\exp(-i\omega t)\rangle$$

$$= \langle\exp(-i\omega t)\alpha|\sqrt{\frac{\hbar}{2m\omega}}\left[a^\dagger + a\right]|\alpha\exp(-i\omega t)\rangle$$

$$= \langle\exp(-i\omega t)\alpha|\sqrt{\frac{\hbar}{2m\omega}}\left[\exp(i\omega t)\alpha^* + \exp(-i\omega t)\alpha\right]|\alpha\exp(-i\omega t)\rangle$$

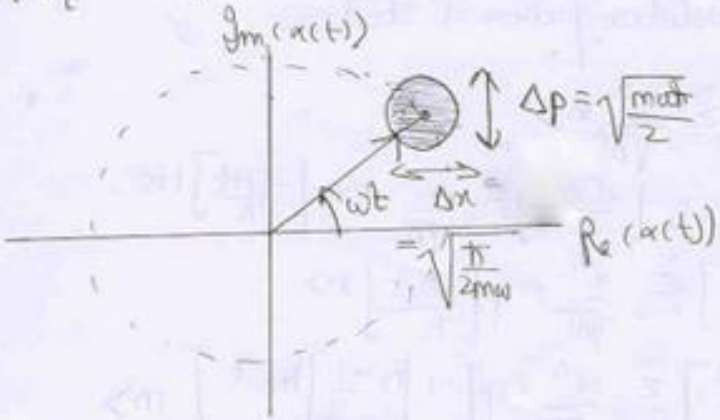
$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\exp(i\omega t)\alpha^* + \exp(-i\omega t)\alpha\right]$$

$\langle\exp(-i\omega t)\alpha|\exp(-i\omega t)\alpha\rangle$
$\Downarrow$
$= 1$ (Normalized)

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\cos(\omega t)(\alpha+\alpha^*) + i\sin(\omega t)(\alpha^*-\alpha)\right]$$

$$\langle \hat{x} \rangle_t = \langle \hat{x} \rangle_0 \cos(\omega t) + \frac{\langle \hat{p} \rangle_0}{m\omega} \sin(\omega t)$$

$$\langle \hat{p} \rangle_t = \langle \hat{p} \rangle_0 \cos(\omega t) - m\omega \langle \hat{x} \rangle_0 \sin(\omega t)$$



- Thus coherent states oscillate in classical phase space with "ω" frequency. Their mean changes but variance is still fixed.
- Also note that length of arrow is indicative of energy

$$\text{Length} = \frac{\langle \hat{p} \rangle_\alpha^2}{2m\omega\hbar} + \frac{m\omega}{2\hbar} \langle \hat{x} \rangle_\alpha^2$$

$$= \frac{1}{\hbar\omega} \left[\frac{\langle \hat{p} \rangle_\alpha^2}{2m} + \frac{1}{2} m\omega^2 \langle \hat{x} \rangle_\alpha^2 \right]$$

$$= \frac{E_\alpha}{\omega\hbar}$$

5. Quantum-Classical Correspondence

Till now we have seen classical behavior of Quantum Coherent States. This means that if the Hamiltonian of our quantum system allows us to write coherent states of the system, we can always understand it in terms of classical aspects.

So let us formalize this model further.

- Transformations and Invariance :-

The most abstract definition of transformation that one can come across is given as:

Given two vector spaces V & W, a transformation T is a map

$T: V \longrightarrow W$.

However, in physics we mostly deal with linear transformations. So let us add some more rules.

A linear transformation between two vector spaces V & W is a map

$T: V \longrightarrow W$ such that

i) $T(v_1 + v_2) = T(v_1) + T(v_2) \quad \forall\, v_1, v_2 \in V$, and

ii) $T(av) = a\,T(v) \quad \forall\, v \in V$ & $a \in F$ (scalar)

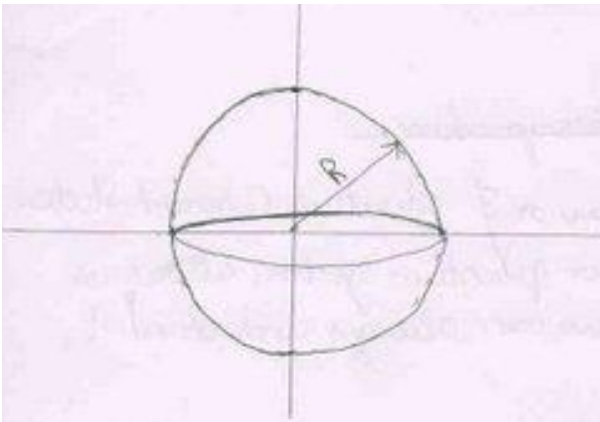
Note that $T(0) = 0$.

- Invariance and Symmetry :-

The next question we have to ask is "Given a transformation T, how do we say, it is a symmetry of the system?"

Well, to answer this, we have to understand the meaning of symmetry itself.

Let us consider sphere as our system. The sphere is perfectly symmetric, isotropic & homogeneous.



Now consider a shell at radius R inside this sphere.
All the points on this shell exhibit "rotational" "Symmetry" in the sense that it leads to a conserved quantity – "Length".

$$L^2 = x^2 + y^2 + z^2 = R^2$$

∴, We expect a symmetry to give us a conserved quantity.

Now let us formalize our physical intuition and involve some mathematics. Consider a quantum system given by Hamiltonian H in Hilbert Space $\mathcal{H}$.

Consider eigen states of H

$$H|\psi\rangle = E_\psi|\psi\rangle$$

Now, let us say we want a symmetry such that energy is conserved. So let T be our symmetry such that

$H\,T|\psi\rangle = E_\psi T|\psi\rangle$ Now, $T|\psi\rangle$ & $|\psi\rangle$ may be degenerate or non degenerate.

In case of degeneracy $T|\psi\rangle$ & $|\psi\rangle$ are different states. In case of nondegeneracy $T|\psi\rangle = |\psi\rangle$

But irrespective of degeneracy or nondegeneracy E_ψ is same as T is symmetry as we demanded.

$$\because \quad HT|\psi\rangle = E_\psi T|\psi\rangle$$

$$\Rightarrow T^\dagger H T|\psi\rangle = E_\psi|\psi\rangle \quad \text{(I considered } T \text{ to be unitary)}$$

$$\therefore T^\dagger H T|\psi\rangle = H|\psi\rangle = E_\psi|\psi\rangle$$

$$\Rightarrow [T^\dagger H T - H]|\psi\rangle = 0$$

$$\Rightarrow T^\dagger H T = H \Rightarrow \boxed{[T, H] = 0}$$

Thus, if T conserves eigenvalues of H, $[T, H] = 0$

- The conserved quantity is nothing but "invariant"!!

· Another implication of above exercise is no matter if we transform the Hamiltonian of system or the state of test particle experiencing the actions, if transformation is a symmetry, we get an invariant, a conserved quantity.

With these Physical and Mathematical tools in our hands, let us move on to our next venture - "Constructing Vector Models".

6. Vector Model of Quantum Mechanics

Given a transformation T and a vector space V. A subspace W of V is said to be invariant under T if

given $T: V \rightarrow V$ $\exists$ restriction T_W:

$$T_W : W \rightarrow W$$

Now, consider Hilbert space $\mathcal{H}_N$. We construct a quantum system that is invariant under transformations of O(N) group where N is number of spatial dimensions.

Now, our Hilbert space $\mathcal{H}_N$ is a space of O(N) invariant wavefunctions. For describing these wavefunctions, we construct O(N) invariant operators. If $\hat{X}$ is position operator & $\hat{P}$ is momentum operator then commutation relation dictates

- $$[\hat{X}_i, \hat{P}_j] = i\hbar\delta_{ij}$$

Let us consider our new operators as

$$\hat{x}_i = \frac{\hat{X}_i}{\sqrt{N}} \quad \& \quad \hat{p}_i = \frac{\hat{P}_i}{\sqrt{N}}$$

$\therefore$ Our new commutation relation is

$$[\hat{x}_i, \hat{p}_j] = \frac{i\hbar}{N}\delta_{ij}$$

- Now, additionally consider that our quantum system has n interacting test "particles". We use α, β to label them.

$\therefore$ Our commutation relation further modifies to

$$[\hat{x}_i(\alpha), \hat{p}_j(\beta)] = \frac{i\hbar}{N}\delta_{ij}\delta_{\alpha\beta} \qquad \begin{matrix} i,j = 1 \rightarrow N \\ \alpha,\beta = 1 \rightarrow n \end{matrix} \qquad (14)$$

Along with this relation, we construct three invariant operators:

$$\hat{A}(\alpha,\beta) = \frac{1}{2}\sum_{i=1}^{N}[\hat{x}_i(\alpha)\hat{x}_i(\beta)] \qquad (15)$$

$$\hat{B}(\alpha,\beta) = \frac{1}{2}\sum_{i=1}^{N}[\hat{x}_i(\alpha)\hat{p}_i(\beta) + \hat{p}_i(\beta)\hat{x}_i(\alpha)] \qquad (16)$$

$$\hat{C}(\alpha,\beta) = \frac{1}{2}\sum_{i=1}^{N}[\hat{p}_i(\alpha)\hat{p}_i(\beta)] \qquad (17)$$

Clearly $\hat{A}(\alpha,\beta)$, $\hat{B}(\alpha,\beta)$, $\hat{C}(\alpha,\beta)$ are $O(N)$ invariant.

Now, we claim that Hamiltonian of such an invariant system will be of the form

$\hat{H}_N = N h[\hat{A}(\alpha,\beta), \hat{B}(\alpha,\beta), \hat{C}(\alpha,\beta)]$ where h is arbitrary polynomial of $\hat{A}, \hat{B}, \hat{C}$ with no explicit N dependence.

Let us test the validity of this claim by taking an example.

This brings us back to our QHO!!

This time let us consider N dimensional problem. Since our $\hat{x}$ & $\hat{p}$ now have an inherent factor of $\frac{1}{\sqrt{N}}$. Therefore our Hamiltonian is:

$$\hat{H} = \frac{N\hat{p}^2}{2m} + \frac{N}{2}k\hat{x}^2 = N\left[\frac{\hat{p}^2}{2m} + \frac{k\hat{x}^2}{2}\right]$$

where $\hat{p}^2 = \sum_{i=1}^{N} \hat{p}_i^2$; $\hat{x}^2 = \sum_{i=1}^{N} \hat{x}_i^2$

Since there is only one oscillator $\therefore \alpha = \beta = 1$

$\therefore \hat{A} = \frac{1}{2}\sum_{i=1}^{N} \hat{x}_i^2$; $\hat{B} = \frac{1}{2}\sum_{i=1}^{N}[\hat{x}_i\hat{p}_i + \hat{p}_i\hat{x}_i]$; $\hat{C} = \frac{1}{2}\sum_{i=1}^{N}\hat{p}_i^2$

$$\therefore \hat{H} = \frac{N}{2}\left[k\hat{A} + \frac{\hat{C}}{m}\right] = N\left[\frac{k\hat{A}}{2} + \frac{\hat{C}}{m}\right] = N h[\hat{A},\hat{C}]$$

– as claimed

Let us pause QHO for a second and derive some commutation relation using 14, 15, 16, 17.

- $[\hat{A},\hat{C}] = \frac{i\hbar}{N}\hat{B}$ – (18)
- $[\hat{A},\hat{B}] = \frac{2i\hbar}{N}\hat{A}$ – (19)
- $[\hat{B},\hat{C}] = \frac{2i\hbar}{N}\hat{C}$ – (20)

In Heisenberg picture for QHO:

$$\frac{d\hat{A}}{dt} = \frac{1}{i\hbar}[\hat{A}(t),\hat{H}] = \frac{1}{i\hbar}\left[\hat{A}(t), \frac{N}{2}\left[k\hat{A} + \frac{\hat{C}}{m}\right]\right]$$

$$= \frac{1}{i\hbar}\frac{N}{2m}[\hat{A},\hat{C}] = \frac{\hat{B}}{2m} \quad \text{– (21)}$$

$$\frac{d\hat{B}}{dt} = \frac{1}{i\hbar}[\hat{B}, \hat{H}] = -k\hat{A} + \frac{\hat{C}}{m} \quad \text{— (22)}$$

$$\frac{d\hat{C}}{dt} = \frac{1}{i\hbar}[\hat{C}, \hat{H}] = -\frac{\hat{B}k}{2} \quad \text{— (23)}$$

Now, if we were to study 21, 22, 23 as classical averages with respect to Quantum state then we certainly need a quantum state at first place to find averages.

Well this is exactly where our coherent states $|\alpha\rangle$ enter!!

Taking average of 21, 22, 23 w.r.t $|\alpha\rangle$ and solving, we get

$$\langle\hat{A}\rangle_{|\alpha\rangle} = -\frac{B}{2m}\sqrt{\frac{m}{k}}\cos\left[\sqrt{\frac{k}{m}}t + \theta\right]$$

$$\langle\hat{B}\rangle_{|\alpha\rangle} = B\sin\left[\sqrt{\frac{k}{m}}t + \theta\right]$$

$$\langle\hat{C}\rangle_{|\alpha\rangle} = \sqrt{\frac{m}{k}}\frac{k}{2}B\cos\left[\sqrt{\frac{k}{m}}t + \theta\right]$$

Writing above equations a little bit more neatly, we have

$$\langle\hat{A}\rangle_{|\alpha\rangle} = -B\sqrt{\frac{1}{4mk}}\cos\left[\sqrt{\frac{k}{m}}t + \theta\right] \quad \text{— (24)}$$

$$\langle\hat{B}\rangle_{|\alpha\rangle} = B\sin\left[\sqrt{\frac{k}{m}}t + \theta\right] \quad \text{— (25)}$$

$$\langle\hat{C}\rangle_{|\alpha\rangle} = B\sqrt{\frac{mk}{2}}\cos\left[\sqrt{\frac{k}{m}}t + \theta\right] \quad \text{— (26)}$$

Here B & θ are two constants that depend on initial condition.

We see 24 & 26 behave as if the were ~~K.E & P.E~~ Potential & Kinetic energy of system.

Hence, we are able to extract some information of quantum system using this model.

8. Conclusion

In this report we have seen that Coherent State formulation of Quantum Systems gives a promising approach to link Quantum Mechanics with Classical Mechanics. However, this approach will not work for Hamiltonians that don't have Coherent State Formulation.

We also saw that a more formal approach can be developed using vector models of O(N) invariant Quantum Systems. The coherent state average of O(N) invariant operators resemble classical behavior for N dimensional Quantum Oscillator.

This work has several implications two of which I will list here:

i) In limits of large quantum numbers Quantum Systems behave classically.

ii) I do expect that Quantum Variances reduce to zero in large N limits that I couldn't show in this report.

iii) Written mathematically above point can be restated as:

$$< \hat{A}\hat{B} > = < \hat{A} > < \hat{B} > + O(\frac{1}{N})$$

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