Non Hermetian Quantum Dynamics of Damped Oscillator

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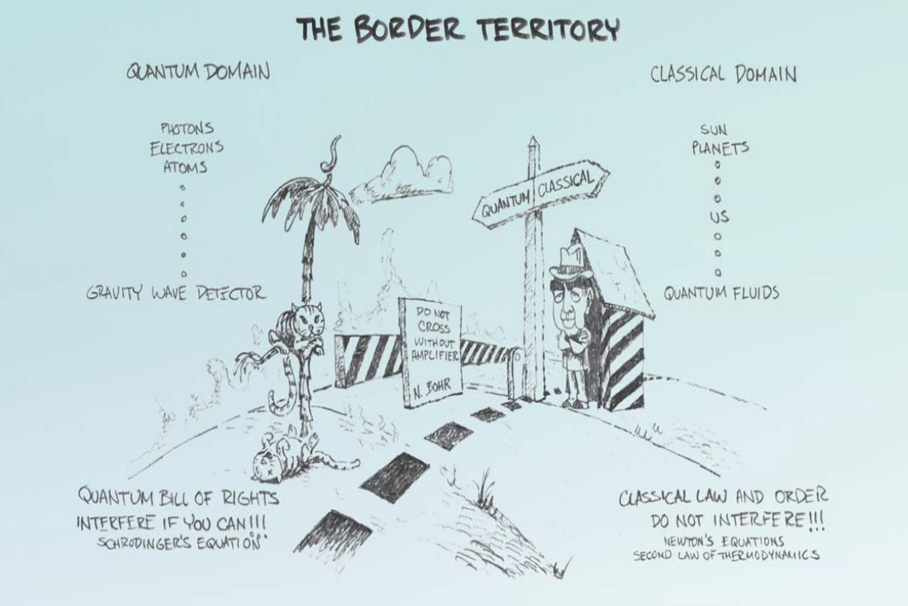
February 17, 2016


THE BORDER TERRITORY
QUANTUM DOMAIN
CLASSICAL DOMAIN
PHOTONS
ELECTRONS
ATOMS
GRAVITY WAVE DETECTOR
SUN
PLANETS
US
QUANTUM FLUIDS
QUANTUM
CLASSICAL
DO NOT CROSS WITHOUT AMPLIFIER
N. BOHR
QUANTUM BILL OF RIGHTS
INTERFERE IF YOU CAN!!!
SCHRODINGER'S EQUATION
CLASSICAL LAW AND ORDER
DO NOT INTERFERE!!!
NEWTON'S EQUATIONS
SECOND LAW OF THERMODYNAMICS

Classical Damped Harmonic Oscillator

- **Classical Equation**

$$\ddot{q} + 2\gamma\dot{q} + \omega^2 q = 0 \tag{1}$$

where,

$\dot{q}$ indicates time derivative

γ is damping coefficient

ω is oscillator frequency

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- **Solution**

$$q(t) = q_o e^{-\gamma t} cos\omega_d t + \left(\frac{\dot{q_o}}{\omega_d} + \frac{\gamma q_o}{\omega_d}\right) e^{-\gamma t} sin\omega_d t \tag{2}$$

where, $\omega_d = \sqrt{\omega^2 - \gamma^2}$ and assume $p_o = m\dot{q_o} + m\gamma q_o$

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- **Quantum Average using Ehrenfest theorem**

$$\langle\alpha, t|\, \hat{q}\, |\alpha, t\rangle = e^{-\gamma t}\left[\langle\alpha|\, \hat{q}\, |\alpha\rangle\, cos\omega_d t + \frac{\langle\alpha|\, \hat{p}\, |\alpha\rangle}{m\omega_d} sin\omega_d t\right] \tag{3}$$

where, $|\alpha, t\rangle$ is coherent state

Quantum Hamiltonian

- **General Non-Hermetian Hamiltonian**

$$\hat{H}_D = \hat{H} - i\hat{\Gamma} \tag{4}$$

where,
$\hat{H} = \hat{H}^\dagger$ is Hamiltonian of undamped system
$\hat{\Gamma} = \hat{\Gamma}^\dagger$ is damping part of Hamiltonian
$[\hat{H}, \hat{\Gamma}] = 0$

Quantum Dynamics

- **Time Evolution**

$$i\hbar\frac{\partial\left|\psi\right\rangle}{\partial t} = \hat{H}_D\left|\psi\right\rangle \tag{5}$$

$$\implies \left|\psi, t+dt\right\rangle = \hat{U}(t, t+dt)\left|\psi, t\right\rangle = \left(1 - i\frac{\hat{H}_D dt}{\hbar}\right)\left|\psi, t\right\rangle \tag{6}$$

where,
$\hat{U}(t, t+dt)$ is non-unitary time evolution operator

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Similarly,

$$\left|\psi, t\right\rangle = \hat{U}(t)\left|\psi, 0\right\rangle = e^{-i\frac{\hat{H}_D t}{\hbar}}\left|\psi, 0\right\rangle \tag{7}$$

Quantum Dynamics

- **Consider a state of damped system**

$$|\psi, t=0\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \tag{8}$$

where, $|n\rangle$ are energy eigen states of undamped Hamiltonian and $\sum_{n=0}^{\infty} |c_n|^2 = 1$

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$$= \sum_{n=0}^{\infty} c_n e^{-\frac{\Gamma_n t}{\hbar}} e^{-i\frac{\hat{E}_n t}{\hbar}} |n\rangle \tag{10}$$

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- **Time Depedent Norm!!**

$$N(t) = \langle\psi, t|\psi, t\rangle = \sum_{n=0}^{\infty} |c_n|^2 e^{-2\frac{\Gamma_n t}{\hbar}} \tag{11}$$

Quantum Dynamics

- **Time Derivative of Norm**

$$\hbar \frac{dN(t)}{dt} = -2 \langle \psi, t | \hat{\Gamma} | \psi, t \rangle \tag{12}$$

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- **Operator Expectation Values**

$$\langle \hat{A} \rangle_t = \frac{\langle \psi, t | \hat{A} | \psi, t \rangle}{N(t)} \tag{13}$$

$$= \langle \phi, t | \hat{A} | \phi, t \rangle \tag{14}$$

where, $|\phi, t\rangle = \frac{|\psi, t\rangle}{\sqrt{N(t)}}$

Quantum Dynamics

- **Evolution of Expectation Values**

$$\langle \psi, t+dt|\, \hat{A}\, |\psi, t+dt\rangle = \langle \psi, t|\, \hat{U}^{\dagger}(dt)\hat{A}\hat{U}(dt)\, |\psi, t\rangle \tag{15}$$

$$\implies i\hbar \frac{d\, \langle \psi, t|\, \hat{A}\, |\psi, t\rangle}{dt} = \langle \psi, t|\, [\hat{H}, \hat{A}]\, |\psi, t\rangle - i\, \langle \psi, t| \left\{\hat{\Gamma}, \hat{A}\right\} |\psi, t\rangle \tag{16}$$

where, $[\hat{H}, \hat{A}] = \hat{H}\hat{A} - \hat{A}\hat{H}$ and $\left\{\hat{\Gamma}, \hat{A}\right\} = \hat{\Gamma}\hat{A} + \hat{A}\hat{\Gamma}$

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where, $[\hat{H}, \hat{A}] = \hat{H}\hat{A} - \hat{A}\hat{H}$ and $\left\{\hat{\Gamma}, \hat{A}\right\} = \hat{\Gamma}\hat{A} + \hat{A}\hat{\Gamma}$

But expectations have to be calculated using normalized states, hence

$$i\hbar\frac{d\,\langle\phi, t|\,\hat{A}\,|\phi, t\rangle}{dt} = \langle\phi, t|\,[\hat{H}, \hat{A}]\,|\phi, t\rangle - 2i\Delta^2_{A\Gamma} \tag{17}$$

where, $\Delta^2_{A\Gamma} = \langle\phi, t|\,\frac{1}{2}\left\{\hat{A}, \hat{\Gamma}\right\}|\phi, t\rangle - \langle\phi, t|\,\hat{A}\,|\phi, t\rangle\,\langle\phi, t|\,\hat{\Gamma}\,|\phi, t\rangle$

Quantum Dynamics

Modified Schrödinger Equation

$$i\hbar\frac{d\left|\phi,t\right\rangle}{dt} = \hat{H}\left|\phi,t\right\rangle - i[\hat{\Gamma},\left|\phi,t\right\rangle\left\langle\phi,t\right|]\left|\phi,t\right\rangle \tag{18}$$

where, $|\phi, t\rangle$ are normalized kets

Coherent States

- **For our case, we choose**

$$\hat{\Gamma} = \hbar\gamma\hat{N} \tag{19}$$

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- **Hence Hamiltonian for DHO**

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$$|\alpha, 0\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{21}$$

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$$|\alpha, t\rangle = \hat{U}(t) |\alpha, 0\rangle = e^{-i \frac{\hat{H}_D t}{\hbar}} |\alpha, 0\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-\gamma n t} e^{-i \frac{\hat{E}_n t}{\hbar}} |n\rangle \tag{22}$$

Coherent States

- **Time Depedent Norm**

$$N(t) = \langle \alpha, t | \alpha, t \rangle = e^{|\alpha^2| e^{-2\gamma t}} \tag{23}$$

Normalized Coherent States

$$|\alpha, t\rangle = e^{-\frac{|\alpha^2|}{2} e^{-2\gamma t}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-\gamma n t} e^{-i \frac{\hat{E}_n t}{\hbar}} |n\rangle \tag{24}$$

Quantum to Classical Transition

- **Position Expectation**

$$
\begin{aligned}
q(t) = \langle \alpha, t | \hat{q} | \alpha, t \rangle &= \langle \alpha, t | \sqrt{\frac{\hbar}{2m\omega_d}} (\hat{a} + \hat{a}^\dagger) | \alpha, t \rangle \\
&= \sqrt{\frac{2\hbar}{m\omega_d}} Re \left(\alpha e^{-it\omega_d} \right) e^{-\gamma t} \qquad (25)
\end{aligned}
$$

Note that, $Re(\alpha) = \sqrt{\frac{m\omega_d}{2\hbar}} q_o$

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- **Momentum Expectation**

$$p(t) = \langle \alpha, t | \, \hat{p} \, | \alpha, t \rangle = \langle \alpha, t | \sqrt{\frac{m\omega_d \hbar}{2}} \left(\frac{\hat{a} - \hat{a}^\dagger}{i} \right) | \alpha, t \rangle$$
$$= \sqrt{2m\omega_d \hbar} \;\; Im \left(\alpha e^{-it\omega_d} \right) e^{-\gamma t} \qquad (26)$$

Note that, $Im(\alpha) = \sqrt{\frac{1}{2m\omega_d \hbar}} p_o$

Quantum to Classical Transition

- **Position Variance**

$$\begin{aligned}\langle \Delta \hat{q}^2 \rangle &= \frac{\hbar}{2m\omega_d} + \frac{\hbar}{m\omega_d}\left[|\alpha|^2 + Re(\alpha^2 e^{-it2\omega_d}) - 2\left(Re(\alpha e^{-it\omega_d})\right)^2\right] e^{-2\gamma t} \\ &= \frac{\hbar}{2m\omega_d} \qquad (27)\end{aligned}$$

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Therefore, $|\alpha, t\rangle$ form minimum uncertainty states.

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Therefore, $|\alpha, t\rangle$ form minimum uncertainty states.

- Heisenberg's Uncertainty is Safe!

$$\langle \alpha, t| \Delta \hat{x}^2 |\alpha, t\rangle \langle \alpha, t| \Delta \hat{p}^2 |\alpha, t\rangle = \frac{\hbar^2}{4} \tag{29}$$

Quantum to Classical Transition

- **Energy Expectation**

$$\begin{aligned}\langle\alpha, t|\,\hat{H}\,|\alpha, t\rangle &= e^{-|\alpha^2|e^{-2\gamma t}}\sum_{n,m=0}^{\infty}\frac{|\alpha|^{2n}}{n!}e^{-2\gamma tn}\left[n+\frac{1}{2}\right]\hbar\omega_d \\ &= |\alpha|^2 e^{-2\gamma t}\hbar\omega_d + \frac{1}{2}\hbar\omega_d \end{aligned} \tag{30}$$

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- **Position Evolution**

$$i\hbar \frac{d\langle \hat{q} \rangle}{dt} = \langle [\hat{H}, \hat{q}] \rangle - 2i\Delta^2_{q\Gamma} \qquad (31)$$

$$\implies \frac{dq(t)}{dt} = \frac{p(t)}{m} - \gamma q(t) \qquad (32)$$

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Differentiate (35) wrt time and use (36) to get,

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but, $\omega_d^2 + \gamma^2 = \omega^2$

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Back to Classical Equation

$$\ddot{q} + 2\gamma\dot{q} + \omega^2 q = 0 \tag{38}$$

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  1. Correct Classical limit
  2. Correct $\gamma \to 0$ limit

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- **Challenges**
  1. Reservoir Coupling
  2. Measure of Entropy

Danke!